2. Let V be an inner product space. Prove that for all $x, y \in V$,

$$\big|\|x\| - \|y\|\big| \le \|x - y\|.$$

1

Poll

Taking the squares, we get $\|x\|^2 + \|y\|^2 - 2\|x\|\cdot\|y\| \le \|x-y\|^2$. Let's prove this!

$$\|x\|^2 + \|y\|^2 - 2\|x\|\cdot\|y\|$$

$$= \langle x, x\rangle + \langle y, y\rangle - 2\|x\|\cdot\|y\|$$

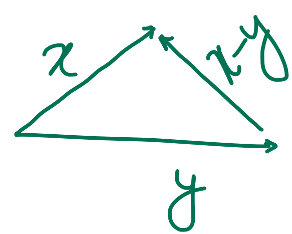
$$\le \langle x, x\rangle + \langle y, y\rangle - 2|\langle x, y\rangle| \quad \text{(CS inequality)}$$

Careful about the negative sign!

$$\le \langle x, x\rangle + \langle y, y\rangle - \langle x, y\rangle - \langle y, x\rangle$$

b/c $2|\langle x, y\rangle| = 2\sqrt{\mathrm{Re}^2 + \mathrm{Im}^2}$
$\ge 2\cdot\mathrm{Re} = \langle x, y\rangle + \langle y, x\rangle$.

$$= \langle x-y, x-y\rangle = \|x-y\|^2.$$



3. Prove the following inequalities.

a) $a\cos\theta + b\sin\theta \le (a^2+b^2)^{1/2}$, for $a, b, \theta \in \mathbb{R}$. 1

Consider $\mathbb{R}^2$ with the standard inner product, namely dot product.

Let $v = (a,b)$ and $w = (\cos\theta, \sin\theta)$. $\sqrt{\cos^2\theta + \sin^2\theta} = 1$.

Then, by CS inequality, we get

$$\underset{a\cdot\cos\theta + b\sin\theta}{\underset{\parallel}{\langle v, w\rangle}} \le \underset{\sqrt{a^2+b^2}}{\underset{\parallel}{\|v\|}} \cdot \overset{\sqrt{\cos^2\theta+\sin^2\theta}=1}{\overset{\parallel}{\|w\|}}$$

$\therefore$ $a\cos\theta + b\sin\theta \le \sqrt{a^2+b^2}$.

1. Let $V = C([0,1])$ be the real vector space of real-valued continuous functions on $[0,1]$. Define an inner product $\langle \cdot, \cdot \rangle$ on V by

2 Poll

$$\langle f, g \rangle = \int_0^1 t f(t) g(t)\, dt.$$

a) Find an orthonormal basis for $W = P_2(\mathbb{R}) \subseteq V$.

We take for granted that ← such $\langle f, g\rangle$ gives an inner prod.

a) We will apply Gram-Schmidt Orthogonalization Process to a basis $\{1, t, t^2\}$ of W. (v_1 v_2 v_3)

The first one remains the same. $x_1 = v_1$

The second will be modified as $t - \frac{\langle 1, t\rangle}{\langle 1, 1\rangle} 1 = t - \frac{\int_0^1 t^2 dt}{\int_0^1 t\,dt} 1 = t - \frac{\frac{1}{3}}{\frac{1}{2}} = t - \frac{2}{3}.$

$x_3 = v_3 - \mathrm{proj}_{\mathrm{Span}\{v_1, v_2\}}(v_3)$

Now, the third one : $t^2 - \frac{\langle t - \frac{2}{3}, t^2\rangle}{\langle t - \frac{2}{3}, t - \frac{2}{3}\rangle}(t - \frac{2}{3}) - \underbrace{\frac{\langle 1, t^2\rangle}{\langle 1, 1\rangle}}_{\frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}} 1$

← Similarly, $\langle 1, t^2\rangle = \frac{1}{4}$

MUST NOT BE $\frac{\langle t, t^2\rangle}{\langle t, t\rangle} t$

- $\langle t - \frac{2}{3}, t^2\rangle = \int_0^1 (t^4 - \frac{2}{3}t^3) dt$

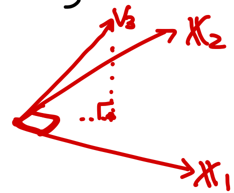
$= 1/5 - 1/4 \cdot 2/3 = \frac{1}{30}$

- $\langle t - \frac{2}{3}, t - \frac{2}{3}\rangle = \int_0^1 (t^3 - \frac{4}{3}t^2 + \frac{4}{9}t) dt$

$= \frac{1}{4} - \frac{4}{3}\cdot\frac{1}{3} + \frac{4}{9}\cdot\frac{1}{2} = \frac{1}{36}$

$$= t^2 - \frac{6}{5}\left(t - \frac{2}{3}\right) - \frac{1}{2} = t^2 - \frac{6}{5}t + \frac{4}{5} - \frac{1}{2} = t^2 - \frac{6}{5}t + \frac{3}{10}.$$

Lastly, normalization!



- $1 \to \frac{1}{\sqrt{\frac{1}{2}}}\, 1 = \sqrt{2}$
- $t - \frac{2}{3} \to \frac{1}{\frac{1}{6}}\left(t - \frac{2}{3}\right) = 6\left(t - \frac{2}{3}\right) = 6t - 4$

- A detour: $\|t^2\|^2 - \|(t-\frac{2}{3})\text{-part}\|^2 - \|1\text{-part}\|^2$

$$= \int_0^1 t^5\,dt - \frac{36}{25}\cdot\left(\frac{1}{6}\right)^2 - \frac{1}{4}\cdot\left(\frac{1}{\sqrt{2}}\right)^2$$

$$= \frac{1}{6} - \frac{1}{25} - \frac{1}{8} = \frac{1}{24} - \frac{1}{25} = \frac{1}{24\cdot 25}$$

$$\therefore \frac{1}{\frac{1}{10\sqrt{6}}}\left(t^2 - \frac{6}{5}t + \frac{3}{10}\right) = \sqrt{6}\left(10t^2 - 12t + 3\right).$$

b) Find the orthogonal projection on W of $f(t) = e^t$.

Given an orthonormal basis,

it's (supposed to be) easy:

$$\mathrm{proj}_W(e^t) = \langle \sqrt{2}, e^t\rangle \sqrt{2} + \langle 6t-4, e^t\rangle (6t-4) + \langle \sqrt{6}(10t^2-12t+3), e^t\rangle \sqrt{6}(10t^2-12t+3).$$

$$= 2\cdot\int_0^1 te^t\,dt + (6t-4)\cdot\int_0^1 (6t^2-4t)e^t\,dt + 6\cdot(10t^2-12t+3)\cdot\int_0^1 (10t^3-12t^2+3t)e^t\,dt$$

$$= 2\cdot 1 + (6t-4)\{6(e-2)-4\} + 6\cdot(10t^2-12t+3)\cdot\{10(-2e+6) - 12(e-2) + 3\}$$

$$= 6\cdot(87-32e)\cdot(10t^2-12t+3) + (6e-16)(6t-4) + 2$$

Note: $\int te^t = (t-1)e^t$ (Integration by parts)

$\int t^2e^t = (t^2-2t+2)e^t$ (")

$\int t^3e^t = (t^3-3t^2+6t-6)e^t$ (")

2. Let V be a finite-dimensional inner product space. Suppose $P \in \mathcal{L}(V,V)$ satisfies $P^2 = P$. Prove that $\ker P = \mathrm{Im}(P)^{\perp}$ if and only if $P = P^*$.

2

P^* is the adjoint operator of P. It is DEFINED to be satisfying $\langle Pv, w\rangle = \langle v, P^* w\rangle$ for all v, w.

($\Longrightarrow$) Suppose that $\ker P = (\mathrm{Im} P)^{\perp}$.

Note that $(I-P)\cdot v \in \ker P$ for any v

b/c $P\cdot(I-P)\cdot v = (P-P^2)v = 0$.

Hence, $\langle Pw, (I-P)\cdot v\rangle = 0$ for any v and w.

$\Rightarrow \langle w, P^*(I-P)\cdot v\rangle = 0$ "

$\Rightarrow P^*(I-P)\cdot v = 0$ for any v $\Rightarrow P^* = P^* P$.

Taking $*$ once more,

$P = P^* P \Rightarrow P^* = P$.

($\Longleftarrow$) In fact, you can go in the reverse way.

If $P^* = P$, then $P^* = P = P^2 = P^* \cdot P \Rightarrow P^*(I-P) = 0$.

$\therefore \langle w, P^*(I-P)v\rangle = 0$ for all v, w. If we restrict v to be in $\ker P$,

$\therefore \langle Pw, (I-P)v\rangle = 0$ " . it proves $\langle Pw, v\rangle = 0$. ∎

2. Let $V = M_{n\times n}(\mathbb{C})$ be equipped with the inner product $\langle A, B\rangle = \operatorname{tr}(B^*A)$. Let P be a fixed invertible matrix in V, and let T_P be the linear operator on V defined by $T_P(A) = P^{-1}AP$. Find the adjoint of T_P.

3

$$\langle T_P A, B\rangle = \langle P^{-1}AP, B\rangle$$

$$= \operatorname{tr}(B^* P^{-1} A P)$$

$$= \operatorname{tr}(P B^* P^{-1} A)$$

$$= \langle A, (PB^*P^{-1})^* \rangle \quad \text{b/c} \quad ** \text{ is the identity.}$$

$$(PB^*P^{-1})^* = (P^{-1})^* B P^*$$

$$\therefore T_P^* B = (P^{-1})^* B P^* = (P^*)^{-1} B P^* = T_{P^*} B.$$

will be proved in class.

1. **(True/False Jeopardy)** Supply convincing reasoning for your answer.

5

(a) T F Every orthogonal set of vectors in an inner product space is linearly independent.

(b) T F Every orthonormal set of vectors in an inner product space is linearly independent.

(c) T F $\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$ is an inner product on $C([-1,1])$ (the real vector space of real-valued continuous functions on $[-1,1]$).

(d) T F If x and y are vectors in an inner product space, then

$$\|x+y\|^2 \le \|x\|^2 + \|y\|^2.$$

(e) T F The product of two self-adjoint operators is always self-adjoint.

(a) False b/c of $\vec{0}$.

(b) True theorem.

(c) False Linearity OR Symmetricity is not the thing.

Strict positivity? Look at the interval.

$$f(x) = \begin{cases} x & \text{on } [-1,0] \\ 0 & \text{on } [0,1] \end{cases}$$

(d) False $x=y$ nonzero vectors

$$\Rightarrow \underbrace{\|2x\|^2}_{= 4\cdot\|x\|^2} \le \underbrace{\|x\|^2+\|x\|^2}_{= 2\cdot\|x\|^2}$$

(e) False $A=A^*$, $B=B^*$

Then, $(AB)^* = B^*A^* = BA$.