2. Let V be a finite-dimensional vector space and let $T : V \to V$ be a linear map with eigenvalue λ. If the dot diagram for the eigenvalue λ consists of (only) two dots in the first row and two dots in the second, show that the "top-to-bottom" approach to finding Jordan canonical form will always succeed.

1 Poll

There are four dots and it ends at the second stage.

$\Rightarrow$ (Denoting $T-\lambda I$ by U, $\dim \ker U = 2$ & $\dim \ker U^k = 4$ $\forall k \geq 2$.)

"top-to-bottom" simply means

① find a basis of $\ker U$ first $\Rightarrow$ there will be two vectors v_1 & v_2.

② Solve $U \cdot v = v_1$, say u_1, and $Uv = v_2$, say u_2 a sol'n. Then, $\{v_1, u_1, v_2, u_2\}$ is a Jordan basis.

Things to check : Solvability of $Uv = v_1$ and $Uv = v_2$.

proof. Consider $\overset{=K_\lambda}{W =} \ker U^2$. $\dim W = 4$ & W is U-invariant. (Why?)

Then, $U^2|_W = 0_{\mathcal{L}(W,W)}$, so $\ker(U|_W) \supseteq \operatorname{im}(U|_W)$.
$\hookrightarrow$ 2-dim'l.

By the dim thm, $\dim \operatorname{im}(U|_W) = 2$. $\ker U = \ker(U|_W) = \operatorname{im}(U|_W) \subseteq \operatorname{im} U$.

1. Decide whether or not the following pairings are inner products.

a) $\langle (x,y),(w,z)\rangle = 2xw+3yz$ on $\mathbb{R}^2$.

2

Remember ① Linear ② Symmetric ③ (Strictly) Positive

c) $\langle p,q\rangle = p(0)q(0)+p(1)q(1)+p(2)q(2)$ on $\mathbb{P}_2(\mathbb{R})$.

a) $\langle (x,y),(w,z)\rangle \overset{?}{=} \langle (w,z),(x,y)\rangle$ 2

$2xw+3yz = 2wx+3zy$

$\langle (x,y),(w,z)+c\cdot(w',z')\rangle \overset{?}{=} \langle (x,y),(w,z)\rangle + c\cdot\langle (x,y),(w',z')\rangle$ 1

$2x(w+cw')+3y(z+cz') = 2xw+3yz + c\cdot(2xw'+3yz')$

$\langle (x,y),(x,y)\rangle \geq 0$ and $=0$ iff $(x,y)=\vec{0}$. 3

$= 2x^2+3y^2 \geq 0$ (✓)

$=0 \iff x=y=0$

$\iff (x,y)=\vec{0}$ ☑

(c) 1 and 2 are easy exercises.

3 $\langle p,p\rangle = p(0)^2+p(1)^2+p(2)^2 \geq 0$ obvious.

$=0 \iff p(0)=0,\ p(1)=0,\ p(2)=0.$

Possible approaches:

- Lagrange interpolation
- $p(x)=ax^2+bx+c$ and plug in
- $0,1,2$ are roots

$\Rightarrow p(x) = \text{sth}\cdot x\cdot(x-1)(x-2)$

$\Rightarrow \text{sth}=0$. (degree)

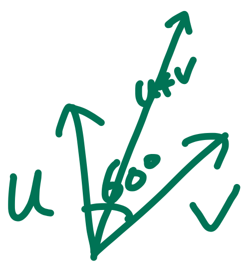
WARNING. (d) is NOT an inner product.

4. Let $u, v \in \mathbb{R}^n$ be unit vectors (their lengths are 1) such that $u \cdot v = 1/2$. Calculate $||u+v||$. **2** Poll

$\hookrightarrow \cos 60° = 1/2$.

$$\|u+v\| = \sqrt{\langle u+v, u+v\rangle} = \sqrt{3}$$

$$\begin{aligned}
\langle u+v, u+v\rangle &= \langle u+v, u\rangle + \langle u+v, v\rangle \\
&= \langle u,u\rangle + \langle v,u\rangle + \langle u,v\rangle + \langle v,v\rangle \\
&= \langle u,u\rangle + 2\cdot\langle u,v\rangle + \langle v,v\rangle \\
&= 1^2 + 2\cdot\tfrac{1}{2} + 1^2 = 3.
\end{aligned}$$



2. For $x, y, z \in \mathbb{R}$, find the maximum possible value of $x + 2y + 2z$ subject to the constraint that $x^2 + y^2 + z^2 = 1$ and find values the achieve this maximum. (Hint: use Cauchy-Schwarz)

2

Cauchy-Schwarz : $\langle u,v\rangle \le \|u\|\cdot\|v\|$ (OR equivalently, $\langle u,v\rangle^2 \le \langle u,u\rangle\cdot\langle v,v\rangle$).

Consider $\mathbb{R}^3$ w/ the standard inner product, namely dot product.

$$\langle (a,b,c),(x,y,z)\rangle = ax+by+cz.$$

Apply CS inequality to $u=(1,2,2)$, $v=(x,y,z)$.

$$\langle u,v\rangle^2 = (x+2y+2z)^2 \le \underbrace{\langle u,u\rangle}_{=1^2+2^2+2^2=9} \cdot \underbrace{\langle v,v\rangle}_{=x^2+y^2+z^2=1}$$

$\therefore\ x+2y+2z \le 3.$

The equality holds iff $(1,2,2) \,//\, (x,y,z) \iff (x,y,z) = r\cdot(1,2,2)$ $(r>0)$

(same direction) $\iff (x,y,z) = \frac{1}{3}(1,2,2)$

3. Using Cauchy-Schwarz, show that if $f : [0, 1] \to \mathbb{R}$ is continuous, then $\left(\int_0^1 f(x)dx\right)^2 \leq \int_0^1 f(x)^2 dx$.

2

Consider V = the vectorspace of continuous functions from $[0,1]$ to $\mathbb{R}$.

Give an inner product as follows:

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx.$$

(For a very rigorous proof, take Math 104 ☺)

Apply CS inequality to u = the constant function 1 and $v = f$.

$$\underbrace{\langle 1, f \rangle^2}_{= \left(\int_0^1 1 \cdot f(x)dx\right)^2} \leq \underbrace{\langle 1, 1 \rangle}_{= \int_0^1 1\,dx = 1} \cdot \underbrace{\langle f, f \rangle}_{= \int_0^1 f(x)^2 dx}$$

3. Let V be a finite-dimensional vector space and let $\langle \cdot, \cdot \rangle$ be an inner product on V. Show that the map $\Phi : V \to V^*$ defined by $\Phi(v)w = \langle v, w \rangle$ is an isomorphism. As a special case (of surjectivity), every linear map $T : \mathbb{R}^n \to \mathbb{R}$ is of the form $T(v) = w \cdot v$ for some fixed $w \in \mathbb{R}^n$.

3

Three things to check: Linearity, one-to-one, onto.

1 $\Phi(v+cv')$? $\Phi(v)+c\cdot\Phi(v')$. How to compare two objects in V^*? By applying vectors in V.

Apply on arbitrary w:

$\langle v+cv', w\rangle \;=\!=\; \langle v,w\rangle + c\cdot\langle v,w'\rangle$

from Linearity of inner products.

2 If $\Phi(v) = 0_{V^*}$, then for any $w \in V$, $\Phi(v)(w) = 0_{V^*}(w) = 0$.

$\Phi(v)(w) = \langle v,w\rangle$

Consider the case $w = v$, then $\langle v,v\rangle = 0 \Rightarrow v = 0$.

3 Given $f \in V^*$, we want v_f st $\langle v_f, w\rangle = f(w)$.

- Use Gram-Schmidt !!! $\leftarrow$ Maybe go back.
- Consider $\ker f$ and find v_f "perpendicular" to $\ker f$.

You could use $\dim V = \dim V^*$ and skip 3.