1. Prove that if $T : V \to V$ is a linear operator on a finite dimensional vector space V, then $\operatorname{rk} T^m = \operatorname{rk} T^{m+1}$ for some $m \geq 1$, and in this case, $\operatorname{rk} T^m = \operatorname{rk} T^{m+k}$ for all $k \geq 0$.

1

Remember that $\operatorname{rk} T^m \geq \operatorname{rk} T^{m+1}$ obviously.

This is because $\operatorname{im} A \geq \operatorname{im} AB$ so that $\operatorname{rk} A \geq \operatorname{rk} AB$ always.

As the codomain of T is V, $\operatorname{rk} T \leq \dim V$.

Hence, $\dim V \geq \operatorname{rk} T \geq \operatorname{rk} T^2 \geq \cdots$ but this cannot be all "$>$" because $\dim V$ is finite. Therefore, $\operatorname{rk} T^m = \operatorname{rk} T^{m+1}$ for some $m \geq 1$.

However, this implies that $\operatorname{im} T^m = \operatorname{im} T^{m+1} =: W$ (denote!).

Then, $T|_{\operatorname{im} T^m} : \operatorname{im} T^m \longrightarrow \operatorname{im} T^{m+1}$ is an isomorphism. (In fact, it is always surjective.)

So, $T^k|_{\operatorname{im} T^m} : \operatorname{im} T^m \longrightarrow \operatorname{im} T^{m+k}$ is an isomorphism too.

$\therefore$ $\operatorname{rk} T^m = \operatorname{rk} T^{m+k}$ for all $k \geq 1$.

You can prove $\operatorname{rk} T^{m+1} = \operatorname{rk} T^{m+2}$ instead.

2. For the following matrices, find a Jordan basis and put the matrix into Jordan canonical form.

1

a) $A = \begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix}$

• $\chi_A(\lambda) = \det(A - \lambda I)$

(2nd col) $= (3-\lambda)\cdot\left[(4-\lambda)(4-\lambda)-1\right]$

$= (3-\lambda)(3-\lambda)(5-\lambda)$

There are two eigenvalues 3 & 5.

$m_a(3) = 2$, $m_a(5) = 1$.

⇓ automatically

$m_g(3) = 1$ or 2, $m_g(5) = 1$.

$A - 3I = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 1 & 0 & 1 \end{pmatrix}$ clearly has $\operatorname{rk}(A-3I) = 1 \Rightarrow \dim E_3 \, (= m_g(3))$ is 2.

∴ it's diagonalizable.

From the obvious relations bet'n columns, we get

$E_3 = \ker(A - 3I) = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

For $A - 5I = \begin{pmatrix} -1 & & 1 \\ 2 & -2 & 2 \\ 1 & & -1 \end{pmatrix}$, we have $E_5 = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$. (No need to find more b/c $m_g(5) = 1$.)

$\times 1 \quad \times 2 \quad \times 1 \Rightarrow$ zero vector

$\beta = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$, $J = \begin{pmatrix} 3 & & \\ & 3 & \\ & & 5 \end{pmatrix}$.

2. For the following matrices, find a Jordan basis and put the matrix into Jordan canonical form.

Poll

b) $B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Corresponds to a vector in $\ker(B-I)^2$ but not in $\ker(B-I)$.

$\cdot\ \chi_B(\lambda) = \det(B - \lambda I)$ (upper Δ) $= (1-\lambda)^3$

There is only one eigenvalue 1. $m_a(1) = 3$.

$m_g(1)$ could be $1, 2, 3$.

$m_g(1) = \dim \ker(B - I) = 3 - \operatorname{rk}(B - I) = 3 - \operatorname{rk}\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 2.$ $\left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$

$\therefore$ There are two Jordan blocks.

$\Downarrow$

$\ker(B - I) = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$

$\ker(B-I)^2 = \ker \mathbb{O} = \mathbb{R}^3 = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$

$(B-I)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$ $\quad B\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$

$\Rightarrow \beta = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \,\Big|\, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$ & $J = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$

$\gamma = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$, $\delta = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\}$ also works. Many possibilities.

$\bullet\ \bullet\ \bullet$: $\ker(A - \lambda I)$

$\bullet\ \bullet$: " $(\ )^2 = E_\lambda$

$\bullet\ \bullet$: " $(\ )^3$

$\bullet$: " $(\ )^4$

1. **(True/False Jeopardy)** Supply convincing reasoning for your answer.

(a) T F Suppose that T is a linear operator on V and that $\beta = \{v_1, \ldots, v_n\}$ is a basis of V such that $[T]_\beta$ is in Jordan canonical form. If $a_1, \ldots, a_n$ are nonzero scalars, then $\beta' = \{a_1 v_1, \ldots, a_n v_n\}$ is a basis of V such that $[T]_{\beta'}$ is in Jordan canonical form.

H (b) T F If T is a linear operator on V with λ as an eigenvalue, then we may write $V = K_\lambda \oplus W$, where W is some T-invariant subspace.

(c) T F Any linear operator on a finite-dimensional vector space whose characteristic polynomial splits has a Jordan canonical form.

(d) T F Every generalized eigenspace of a linear operator T is T-cyclic.

(a) False.

$T = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$ in $\{e_1, e_2\}$.

Then, in $\{e_1, 2e_2\}$, it becomes $\begin{pmatrix} 1 & 2 \\ & 1 \end{pmatrix}$.

(b) True.

Remember!

$V = \ker(T - \lambda I)^m \oplus \operatorname{Im}(T - \lambda I)^m$

only for an eigenvalue λ and the stabilizing exponent m.

(c) True.

Theorem learned from class.

Only need to think about the case $\chi_A(\lambda) = (\lambda - \lambda_0)^m$ and $m_g(\lambda_0) = 1$.

It contains a basis whose matrix is of the form $\begin{pmatrix} \lambda_0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_0 \end{pmatrix}$.

(d) False.

$T = I_V$, $\lambda = 1$.