

1. Let $T : \mathbb{C}^4 \rightarrow \mathbb{C}^4$ be linear and suppose that $p(T) = 0$ where p is a polynomial of degree 3.

Show that \mathbb{C}^4 is not T -cyclic.

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T -cyclic means the T -cyclic subspace generated by a vector v .

$$= \text{Span}\{v, Tv, T^2v, T^3v, \dots\}$$

$$p(t) = t^4$$

$$T = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \Rightarrow T^4 = 0_{4 \times 4}$$

$$= \text{Span}\{v, Tv, \dots, T^m v\}$$

if $T^{m+1}v$ is a linear combination of $\{v, Tv, \dots, T^m v\}$.

then T -cyclic subspace spanned by $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \mathbb{C}^4$

Guess & Claim: The dimension is not enough. To be precise,

$T^{2+1} \cdot v$ is a linear combination of $\{v, Tv, T^2v\}$.

proof. Let $p(t)$ be $a_0 + a_1 t + a_2 t^2 + a_3 t^3$ with $a_3 \neq 0$ ($\because \deg p = 3$).

Then, $a_3 T^3 = -a_2 T^2 - a_1 T - a_0 I$ and we can divide by a_3 .

$$\therefore T^3 = -\frac{a_2}{a_3} T^2 - \frac{a_1}{a_3} T - \frac{a_0}{a_3} I$$

$$\therefore T^3 v = -\frac{a_2}{a_3} \cdot T^2 v - \frac{a_1}{a_3} \cdot T v - \frac{a_0}{a_3} \cdot v.$$

□

2. Suppose that the eigenvalues of $T \in \mathcal{L}(\mathbb{C}^4, \mathbb{C}^4)$ are 2 and 3 only. Find all possible Jordan canonical forms of T . Don't list two Jordan forms if one can be obtained from the other by changing the order of the Jordan blocks.

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There are four eigenvalues counting the (algebraic) multiplicities.

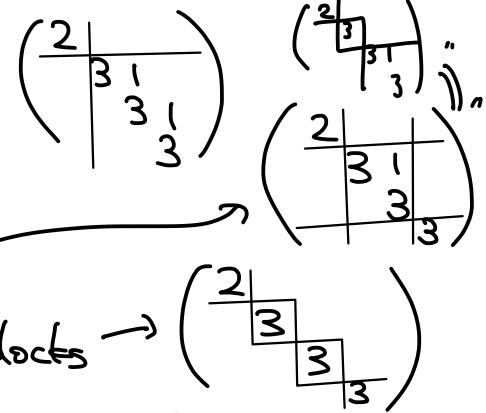
Case 1. One 2, three 3. Case 2. Two 2, two 3. Case 3. Three 2, one 3.

① Only one 2, no variations! (b/c it must be 1×1 J.B.)

Three 3 : - 3×3 Jordan block

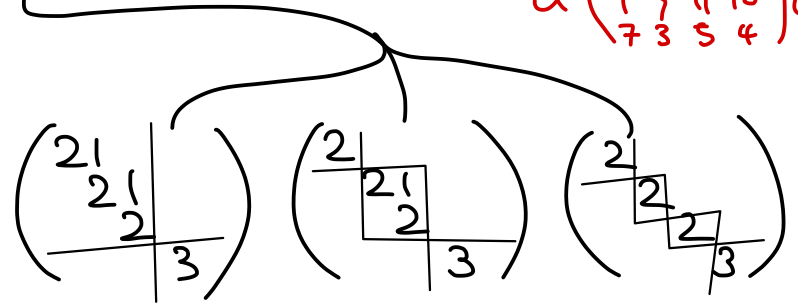
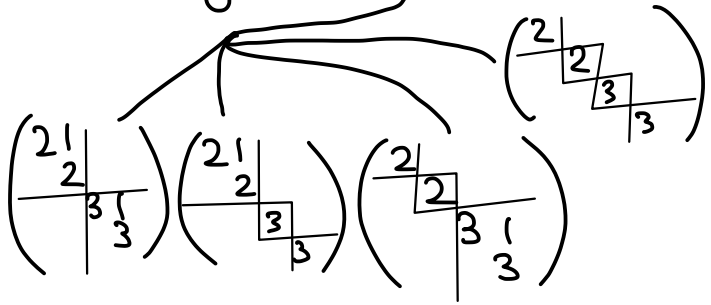
• 2×2 and 1×1 Jordan blocks

• 1×1 and 1×1 and 1×1 Jordan blocks



Similarly for ② and ③,

$Q \begin{pmatrix} 2 & 7 & & \\ & 6 & 2 & \\ & 9 & 11 & 10 \\ & 7 & 3 & 5 & 4 \end{pmatrix} Q^{-1} = \text{one of these 6 possibilities}$



1. Let $T, S \in \mathcal{L}(V, V)$ be commuting linear operators, i.e. $TS = ST$. Show that the generalized eigenspaces $G_\lambda(T)$ are S -invariant.

Definition. $G_\lambda(T) = \{v \in V : (T - \lambda I)^k v = 0 \text{ for some } k\} = K_\lambda(T) \dots$

To prove that W is S -invariant, we prove that

for any $w \in W$, $Sw \in W$. Let's do the same game!

Let w be an arbitrary element of $G_\lambda(T)$. (This is to say $(T - \lambda I)^k w = 0$ for some k .)

We need to prove that Sw satisfies $(T - \lambda I)^l Sw = 0$ for some l .

However, from $ST = TS$, we get $(T - \lambda I)S = TS - \lambda S = ST - S\lambda = S(T - \lambda I)$.

Inductively, we get $(T - \lambda I)^m S = S \cdot (T - \lambda I)^m$ and so,

$$(T - \lambda I)^k Sw = S \cdot (T - \lambda I)^k w = S \cdot 0 = 0.$$

THE k from above



1. Show that Jordan blocks are always similar to their transposes. Conclude that A is similar to A^t for any $A \in M_{n \times n}(\mathbb{C})$.

$$J = QJ^tQ^{-1} \text{ for } Q = (, \dots,).$$

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Any Jordan block J is $\lambda I + N$ where $N = \begin{pmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{pmatrix}$.

$$\begin{aligned} \text{So, for any invertible } Q, \quad QJQ^{-1} &= Q(\lambda I + N)Q^{-1} \\ &= \lambda QIQ^{-1} + QNQ^{-1} = \lambda I + QNQ^{-1}. \end{aligned}$$

So, we need to find Q s.t. $QNQ^{-1} = N^t$.

But, N behaves like $e_n \rightarrow e_{n-1} \rightarrow \dots \rightarrow e_1 \rightarrow 0$. $N^t: e_1 \rightarrow e_2 \rightarrow \dots \rightarrow e_n \rightarrow 0$.

So, $Q =$ the change of coordinates matrix $\{e_n, \dots, e_1\} \rightarrow \{e_1, \dots, e_n\}$ will work.
 $= (, \dots,)$. You can do sanity check!

Note that similarity is transitive & $B = QAQ^{-1} \Rightarrow B^t = (Q^t)^{-1}A^tQ^t$
that is, if $A \sim B$, then $A^t \sim B^t$.

Let J be a Jordan canonical form of A .

$$\text{Then, } A \sim J \sim J^t \sim A^t.$$

1. (True/False Jeopardy) Supply convincing reasoning for your answer.

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- (a) T F Reordering the elements of a Jordan basis gives another Jordan basis. (A Jordan basis of a linear operator is a basis that puts it into Jordan canonical form.)
- (b) T F If V is a finite-dimensional vector space over \mathbb{C} , then every linear operator on V can be put into Jordan canonical form.
- (c) T F If A and B are both Jordan normal forms for a linear operator T , then $A = B$.
- (d) T F If $T : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is linear and \mathbb{C}^n is T -cyclic, then the Jordan canonical form of T has a single block.

a. False

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ in } \{e_2, e_1\} \text{ is } \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

b. True

This is the essence of LA!

c. False

$$\begin{pmatrix} 1 & \\ & 2 \end{pmatrix} \neq \begin{pmatrix} 2 & \\ & 1 \end{pmatrix}$$

d. False

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

$$\text{char}_T(\lambda) = (\lambda - 1)(\lambda + 1).$$

{ but if T has only one eigenvalue, then it would be true.