1. Let $T : \mathbb{C}^4 \to \mathbb{C}^4$ be linear and suppose that $p(T) = 0$ where p is a polynomial of degree 3. Show that $\mathbb{C}^4$ is not T-cyclic.

1 Poll

T-cyclic means the T-cyclic subspace generated by a vector v.

$$= \operatorname{Span}\{v, Tv, T^2v, T^3v, \cdots\}$$

$$= \operatorname{Span}\{v, Tv, \cdots, T^m v\}$$

if $T^{m+1}v$ is a linear combination of $\{v, Tv, \cdots, T^m v\}$.

$p(t) = t^4$

$T = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} \Rightarrow T^4 = O_{4\times4}$

then T-cyclic subsp genid by $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \mathbb{C}^4$

Guess & Claim : The dimension is not enough. To be precise,

$T^{2+1} \cdot v$ is a linear combination of $\{v, Tv, T^2 v\}$.

proof. Let $p(t)$ be $a_0 + a_1 t + a_2 t^2 + a_3 t^3$ with $a_3 \neq 0$ ($\because \deg p = 3$).

Then, $a_3 T^3 = -a_2 T^2 - a_1 T - a_0 \cdot I$ and we can divide by a_3.

$$\therefore T^3 = -\frac{a_2}{a_3} T^2 - \frac{a_1}{a_3} T - \frac{a_0}{a_3} I$$

$$\therefore T^3 v = -\frac{a_2}{a_3} \cdot T^2 v - \frac{a_1}{a_3} \cdot Tv - \frac{a_0}{a_3} \cdot v.$$

$\blacksquare$

2. Suppose that the eigenvalues of $T \in \mathcal{L}(\mathbb{C}^4, \mathbb{C}^4)$ are 2 and 3 only. Find all possible Jordan canonical forms of T. Don't list two Jordan forms if one can be obtained from the other by changing the order of the Jordan blocks.

2 Poll

There are four eigenvalues counting the (algebraic) multiplicities.

Case 1. One 2, three 3. Case 2. Two 2, two 3. Case 3. Three 2, one 3.

① Only one 2, no variations! (b/c it must be) 1×1 J.6.

Three 3 : • 3×3 Jordan block $\rightarrow$ $\begin{pmatrix} 2 & & & \\ & 3 & 1 & \\ & & 3 & 1 \\ & & & 3 \end{pmatrix}$

• 2×2 and 1×1 Jordan blocks $\rightarrow$ $\begin{pmatrix} 2 & & & \\ & 3 & 1 & \\ & & 3 & \\ & & & 3 \end{pmatrix}$

• 1×1 and 1×1 and 1×1 Jordan blocks $\rightarrow$ $\begin{pmatrix} 2 & & & \\ & 3 & & \\ & & 3 & \\ & & & 3 \end{pmatrix}$

$\begin{pmatrix} 2 & & & \\ & 3 & & \\ & & 3 & 1 \\ & & & 3 \end{pmatrix}$ ""

Similarly for ② and ③,

②:

$\begin{pmatrix} 2 & 1 & & \\ & 2 & & \\ & & 3 & 1 \\ & & & 3 \end{pmatrix}$ $\begin{pmatrix} 2 & 1 & & \\ & 2 & & \\ & & 3 & \\ & & & 3 \end{pmatrix}$ $\begin{pmatrix} 2 & & & \\ & 2 & & \\ & & 3 & 1 \\ & & & 3 \end{pmatrix}$ $\begin{pmatrix} 2 & & & \\ & 2 & & \\ & & 3 & \\ & & & 3 \end{pmatrix}$

③:

$\begin{pmatrix} 2 & 1 & & \\ & 2 & 1 & \\ & & 2 & \\ & & & 3 \end{pmatrix}$ $\begin{pmatrix} 2 & & & \\ & 2 & 1 & \\ & & 2 & \\ & & & 3 \end{pmatrix}$ $\begin{pmatrix} 2 & & & \\ & 2 & & \\ & & 2 & \\ & & & 3 \end{pmatrix}$

$Q\begin{pmatrix} 2 & 7 & & \\ & 10 & & 2 \\ 1 & 9 & 11 & 10 \\ 7 & 3 & 5 & 4 \end{pmatrix}Q^{-1}$ = one of these 10 possibilities

1. Let $T, S \in \mathcal{L}(V, V)$ be commuting linear operators, i.e. $TS = ST$. Show that the generalized eigenspaces $G_\lambda(T)$ are S-invariant.

3

Definition. $G_\lambda(T) = \{v \in V : (T-\lambda I)^k v = 0 \text{ for some } k\}$ $= K_\lambda(T) \ldots$

To prove that W is S-invariant, we prove that for any $w \in W$, $Sw \in W$. Let's do the same game!

Let w be an arbitrary element of $G_\lambda(T)$. (This is to say $(T-\lambda I)^k w = 0$ for some k.)

We need to prove that $S \cdot w$ satisfies $(T-\lambda I)^\ell Sw = 0$ for some ℓ.

However, from $ST = TS$, we get $(T-\lambda I)S = TS - \lambda S = ST - S\lambda$
$= S(T-\lambda I)$.

Inductively, we get $(T-\lambda I)^m S = S \cdot (T-\lambda I)^m$ and so,

$$\underbrace{(T-\lambda I)^k}_{\text{THE } k \text{ from above}} Sw = S \cdot (T-\lambda I)^k w = S \cdot 0 = 0.$$

$\blacksquare$

1. Show that Jordan blocks are always similar to their transposes. Conclude that A is similar to A^t for any $A \in M_{n\times n}(\mathbb{C})$.

$J = QJ^tQ^{-1}$ for $Q = \begin{pmatrix} & & 1 \\ & ⋰ & \\ 1 & & \end{pmatrix}$. 4

Any Jordan block J is $\lambda I + N$ where $N = \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$.

So, for any invertible Q, $QJQ^{-1} = Q(\lambda I + N)Q^{-1}$

$= \lambda \cdot QIQ^{-1} + QNQ^{-1} = \lambda I + QNQ^{-1}$.

So, we need to find Q s.t. $QNQ^{-1} = N^t$.

But, N behaves like $e_n \to e_{n-1} \to \cdots \to e_1 \to 0$. N^t: $e_1 \to e_2 \to \cdots \to e_n \to 0$.

So, Q = the change of coordinates matrix $\{e_n, \cdots, e_1\} \to \{e_1, \cdots, e_n\}$ will work.

$= \begin{pmatrix} & & 1 \\ & ⋰ & \\ 1 & & \end{pmatrix}$. You can do sanity check!

Note that similarity is transitive & $B = QAQ^{-1} \Rightarrow B^t = (Q^t)^{-1}A^tQ^t$, that is, if $A \sim B$, then $A^t \sim B^t$.

Let J be a Jordan canonical form of A.

Then, $A \sim J \sim J^t \sim A^t$.

1. **(True/False Jeopardy)** Supply convincing reasoning for your answer.

5

(a) T F Reordering the elements of a Jordan basis gives another Jordan basis. (A Jordan basis of a linear operator is a basis that puts it into Jordan canonical form.)

(b) T F If V is a finite-dimensional vector space over $\mathbb{C}$, then every linear operator on V can be put into Jordan canonical form.

(c) T F If A and B are both Jordan normal forms for a linear operator T, then $A = B$.

(d) T F If $T : \mathbb{C}^n \to \mathbb{C}^n$ is linear and $\mathbb{C}^n$ is T-cyclic, then the Jordan canonical form of T has a single block.

a. False

$\begin{pmatrix}0&1\\0&0\end{pmatrix}$ in $\{e_2, e_1\}$ is $\begin{pmatrix}0&0\\1&0\end{pmatrix}$.

b. True

This is the essence of LA!

c. False

$$\begin{pmatrix}1&\\&2\end{pmatrix} \neq \begin{pmatrix}2&\\&1\end{pmatrix}$$

d. False

$T=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ and $v=\begin{pmatrix}0\\1\end{pmatrix}$.

$\mathrm{char}_T(\lambda)=(\lambda-1)(\lambda+1)$.

but if T has only one eigenvalue, then it would be true.