2. Let $T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ be defined by $T(f(x)) = f'(x) + f''(x)$. Determine whether T is diagonalizable, and if so, find a basis β for $P_3(\mathbb{R})$ such that $[T]_\beta$ is a diagonal matrix.

1

- Diagonalizability $\longleftrightarrow$ Existence of eigenbasis

- $\underbrace{T(f(x))}_{f'(x)+f''(x)} = \overset{\text{eigenvalue}}{c} \cdot \underset{\text{eigenvector}}{f(x)}$ Comparing the degree, we have $c=0$.

  Why don't we care $f(x)=0$ case? Poll.

- $f'(x) = -f''(x) \Rightarrow$ Again comparing the degree, we get $f'(x)=0 \Rightarrow f(x)=d$.

- There is only ~~1~~ one -dim'l eigenspace, so the linear transformation is NOT diagonalizable.

$$\sum m_g(\lambda_i) = 1 < 4.$$

4. Let T be an invertible linear operator on a finite dimensional vector space V. Prove that if T is diagonalizable, then T^{-1} is diagonalizable.

Implies $\lambda \neq 0$.

1 Poll

- Diagonalizability $\longleftrightarrow$ Existence of eigenbasis

$$T = QDQ^{-1}$$
$$\Downarrow (\ )^{-1}$$
$$T^{-1} = QD^{-1}Q^{-1}$$
$$\downarrow$$
diagonal

- If $0 \neq v$ satisfies $Tv = \lambda v$ for some λ, then

$$T^{-1}Tv = T^{-1}\lambda v = \lambda \cdot T^{-1}v$$

$$\Rightarrow \quad \lambda^{-1} \cdot v = T^{-1}v \qquad \therefore v : \text{eigenvector for } T^{-1}.$$

$T^2, T^3 \ldots$

$(T^2v = \lambda^2 v)$

$Tv = \lambda v$

$\Downarrow$

$(T^2+T)v = (\lambda^2+\lambda)\cdot v$

- Suppose $\{v_1, \cdots, v_n\}$ is an eigenbasis for T.

Then, $\{v_1, \cdots, v_n\}$ is a basis + eigenvectors for T

$\Downarrow$

" " T^{-1}

$\downarrow$

$T^2, T^3, \ldots$

Hence, $\{v_1, \cdots, v_n\}$ is an eigenbasis for T^{-1}!

$\downarrow$

$T^2, T^3, \ldots$

2. Find the general solution to the following system:

2

$$\begin{aligned} x_1'(t) &= -x_1(t) - x_2(t) + 3x_3(t) \\ x_2'(t) &= x_1(t) + x_2(t) - x_3(t) \\ x_3'(t) &= -x_1(t) - x_2(t) + 3x_3(t). \end{aligned}$$

$$\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}' = \overbrace{\begin{pmatrix}-1&-1&3\\1&1&-1\\-1&-1&3\end{pmatrix}}^{A\,=}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$

$$\begin{pmatrix}1&1&1\\0&1&-1\\1&1&0\end{pmatrix}^{-1}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}c_1e^{2t}\\c_2e^{t}\\c_3\end{pmatrix}$$

$$\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}c_1e^{2t}+c_2e^{t}+c_3\\c_2e^{t}-c_3\\c_1e^{2t}+c_2e^{t}\end{pmatrix}$$

where $c_1, c_2, c_3 \in \mathbb{R}$.

- A is not invertible, $\dim \operatorname{im} A = 2$
  $\Rightarrow \dim \ker(A - 0\cdot I) = 1$. $\ni \begin{pmatrix}1\\-1\\0\end{pmatrix}$
- $\operatorname{tr} A =$ sum of eigenvalues $= 3$.
- row entries add up to 1.
  $\Rightarrow \begin{pmatrix}1\\1\\1\end{pmatrix}$ gives an eigenvalue 1.
- the left one is 2. eigenvector $= \begin{pmatrix}1\\0\\1\end{pmatrix}$.

$$A = \begin{pmatrix}1&1&1\\0&1&-1\\1&1&0\end{pmatrix}\begin{pmatrix}2&&\\&1&\\&&0\end{pmatrix}\begin{pmatrix}&&\\&&\\&&\end{pmatrix}^{-1}$$

1. If v_1 and v_2 are eigenvectors of a linear operator T with eigenvalues λ_1 and λ_2 and $\lambda_1 \neq \lambda_2$, show that the T-cyclic subspace generated by $v_1 + v_2$ is 2-dimensional.

3

T-cyclic subspace generated by $v = \mathrm{Span}\{v, Tv, T^2v, \dots\}$

For $v = v_1 + v_2$, $T(v) = \lambda_1 v_1 + \lambda_2 v_2 \in \mathrm{Span}\{v_1, v_2\}$.

Induction maybe.

In fact, $T(v_1) = \lambda_1 v_1 \in \mathrm{Span}\{v_1, v_2\}$ $\Rightarrow T^n(v_1+v_2) \in \mathrm{Span}\{v_1, v_2\}$

$T(v_2) = \lambda_2 v_2 \in$ " $\Rightarrow$ our space $\subseteq \mathrm{Span}\{v_1, v_2\}$.

Moreover, $\{v_1+v_2, T(v_1+v_2)\}$ is linearly independent b/c

$c_1(v_1+v_2) + c_2(\lambda_1 v_1 + \lambda_2 v_2) = 0 \Rightarrow \begin{matrix} c_1 + c_2\lambda_1 = 0 \\ c_1 + c_2\lambda_2 = 0 \end{matrix}$ and $\lambda_1 \neq \lambda_2 \Rightarrow c_1 = c_2 = 0$.

v_1, v_2: L.I. b/c they correspond to distinct e.val's.

$\therefore$ It's 2-dim'l.

2. Let V be a finite-dimensional space and let $T, U \in \mathcal{L}(V, V)$. Suppose further that V is a T-cyclic subspace of itself. Show that $TU = UT$ if and only if $U = g(T)$ for some polynomial $g(t)$.

3

What is the candidate g for $U = g(T)$?

Only thing given is one vector v s.t. $V = \text{Span}\{v, Tv, \dots, T^{n-1}v\}$.

Multiplying v to U: $Uv \in V$

Candidate.

$\therefore \exists\, a_0, \dots, a_{n-1} \in F$ s.t. $Uv = a_0 v + a_1 Tv + \cdots + a_{n-1} T^{n-1} v = g(T) \cdot v$.

Claim. $(U - g(T)) \cdot w = 0$ for all $w \in V$.

Pf. $w \in V$ can be expressed as

$c_0 v + c_1 Tv + \cdots + c_{n-1} T^{n-1} v = f(T) \cdot v$

where $f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-1} x^{n-1}$

But, $f(T) \cdot (U - g(T)) \cdot v = 0$ from the def'n of g.

and $f(T) \cdot U = U \cdot f(T)$ b/c $TU = UT$. $\Rightarrow (U - g(T)) \cdot f(T) \cdot v = 0$ $\therefore (U - g(T)) \cdot w = 0$ ☐

Hence, $U - g(T) = 0_{\mathcal{L}(V,V)}$

$\therefore U = g(T)$.

1. **(True/False Jeopardy)** Supply convincing reasoning for your answer.

(a) T F Suppose that $T : V \to V$ is a linear transformation, and that $\dim V < \infty$. Suppose that $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of T. If $m_a(\lambda_1) + \cdots + m_a(\lambda_n) = \dim V$, then T is diagonalizable. & $m_a(\lambda_i) = m_g(\lambda_i)$.

(b) T F If a linear operator T is diagonalizable, it must have distinct eigenvalues.

(c) T F If $T : V \to V$ is a linear transformation and $g(t)$ is a polynomial such that $g(T) = 0$, then the characteristic polynomial of T divides g.

(d) T F Let W_1 and W_2 be the T-cyclic subspaces generated by v_1 and v_2 respectively. If $v_1 \in W_2$, then $W_1 \subseteq W_2$.

(a) False. $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$

$\dim\ker\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 2 - 1 = 1 \neq 2$.

(b) False. $I_V : V \to V$

(c) False. I_V, $g(t) = t - 1$.

$\chi_{I_V} = (t-1)^n \quad g(I) = I - I = 0$.

(d) True

$v_1 \in W_2$ implies $v_1 =$ lin. combo. of $v_2, Tv_2, \cdots$

$\Rightarrow Tv_1 =$ " of $Tv_2, T^2v_2, \cdots$

$$\underbrace{\operatorname{Span}\{v_1, Tv_1, \cdots\}}_{= W_1} \subseteq \underbrace{\operatorname{Span}\{v_2, Tv_2, \cdots\}}_{= W_2}$$

$v_1 \in W_2$

$Tv_1 \in T(W_2) \subseteq W_2$

$T(W_2) = T(\operatorname{span}\{v_2, Tv_2, T^2v_2, \ldots\})$

$= \operatorname{Span}\{Tv_2, T(Tv_2), \ldots\}$

$\subseteq W_2$

2. Suppose that V is a finite dimensional vector space, and that $T : V \to V$ is a nilpotent linear transformation; i.e., $T^n = 0$ for some $n > 0$. Prove that if T is nonzero, it cannot be diagonalizable.

4

Proof by Contradiction: Suppose T is diagonalizable.

$\exists$ an eigenbasis $\{v_1, \ldots, v_n\}$.

However, $T \cdot v_i = \lambda_i \cdot v_i$ gives $\underbrace{T^n \cdot v_i}_{0} = \lambda_i^n \cdot v_i$ $\therefore \lambda_i = 0$ as v_i is nonzero.

This implies that $T \cdot v_i = 0$ for all $i = 1, \ldots, n$.

$$T \cdot (\underbrace{\text{lin. comb. of } v_i}_{\text{all the vectors in } V}) = 0 \quad \Rightarrow T \text{ is zero.}$$

So, we get a contradiction and so T is NOT diagonalizable.