2. Find all eigenvectors and eigenvalues for the linear map $T : \mathbb{C}^4 \to \mathbb{C}^4$ given by $T(w, x, y, z) = (x, y, z, w)$. Hint: it is not necessary to write down the matrix for T.

1 Pd((

What is the very definition of eigen xxx?

$$T(w, x, y, z) = \lambda \cdot (w, x, y, z).$$
$$\| $$
$$(x, y, z, w)$$

$x = \lambda w$
$y = \lambda x$
$z = \lambda y$
$w = \lambda z$

Taking the product, we get:

$$xyzw = \lambda^4 xyzw.$$

There are two cases now: $\lambda^4 = 1$ OR $xyzw = 0$.

$\lambda^4 - 1 = 0 \Rightarrow (\lambda^2 - 1)(\lambda^2 + 1) = 0$

$\lambda = 1, -1, i, -i$

Let $w = 1$. Then the others are determined automatically.

For λ, you get $(1, \lambda, \lambda^2, \lambda^3)$.

λ	1	-1	i	$-i$
v	$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ i \\ -1 \\ -i \end{pmatrix}$	$\begin{pmatrix} 1 \\ -i \\ -1 \\ i \end{pmatrix}$

if one of them is zero,

then the others are all zeros. (Inductively)

$\Rightarrow (w, x, y, z) = (0, 0, 0, 0)$

We do not count such vectors!

3. Show that the list of functions $\{\sin(x), \sin(2x), \sin(3x), e^x, e^{2x}, e^{3x}\}$ is linearly independent by realizing these functions as eigenvectors of a certain linear transformation.

1 Poll

We learned: Suppose we have $T: V \to V$ a linear transformation.
If $v_1, \cdots, v_n$ are eigenvectors with distinct eigenvalues, then they must be linearly independent.

Converse?

$I_{n\times n} v = 1 \cdot v$.

Idea: eigenvector is something that is "not changing" under the map.

Candidate 1: $T(f) = f'$. (Already know that it's linear.)

$T(e^{2x}) = (e^{2x})' = 2 \cdot e^{2x} \Rightarrow \lambda = 2$ is the eigenvalue. Is this working?

How do you deal with sine functions?

$V = \text{Span}\{$ those 6 fns & cosine fns $\}$.

Candidate 2: $T(f) = f''$ (It's linear.)

$T(\sin 2x) = (\sin 2x)'' = (2\cos 2x)' = -4\sin 2x \Rightarrow \lambda = -4$ is the eigenvalue.

As $-1, -4, -9, 1, 4, 9$ are all distinct, they are linearly independent!

(To be precise, you should specify the domain and codomain.) dom = codom = v.s of diff. fns

3. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a diagonalizable linear transformation. Show that there exists a linear map $S : \mathbb{R}^3 \to \mathbb{R}^3$ such that $S^3 = T$.

2

Suppose that T becomes a diagonal matrix in a basis β.

$$[T]_\beta = \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix}.$$

What would be an "educated guess"?

Something like $\begin{pmatrix} a^{1/3} & & \\ & b^{1/3} & \\ & & c^{1/3} \end{pmatrix}$! (Note that $y = x^3$ is a bijective function from $\mathbb{R}$ to $\mathbb{R}$. Hence, unique $1/3$-power!)

Let S be **the** linear transformation whose matrix in the basis β is $\begin{pmatrix} a^{1/3} & & \\ & b^{1/3} & \\ & & c^{1/3} \end{pmatrix}$

Then, $$[S^3]_\beta = [S]_\beta [S]_\beta [S]_\beta = \begin{pmatrix} a^{1/3} & & \\ & b^{1/3} & \\ & & c^{1/3} \end{pmatrix}^3 = \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} = [T]_\beta.$$

Therefore, $S^3 = T$.

1. Suppose V is finite-dimensional and $T : V \to V$ is linear and $T^2 = I_V$. Show that T is diagonalizable.

3

Some doodling: $T^2 = I_V$. Can you find some info about eigenvalues?

If $Tv = \lambda v$, then $\underset{v = I_V \cdot v}{\overset{\|}{T^2 v}} = TTv = T\lambda v = \lambda T v = \lambda^2 v \Rightarrow \lambda^2 = 1$

$\Rightarrow \lambda = \pm 1$.

We found: ONLY two eigenvalues are possible! Let's compute $\dim E_1 + \dim E_{-1}$ with the hope that it will reach to $\dim V$!

But, no info at all ☹

We have dimension theorem ☺ $\Longrightarrow$

$\dim \ker(T - I_V) + \dim \operatorname{im}(T - I_V) = \dim V$

$\dim \ker(T + I_V) + \dim \operatorname{im}(T + I_V) = \dim V$

Next, $0 = T^2 - I_V = (T - I_V)\cdot(T + I_V)$ $\therefore \operatorname{im}(T + I_V) \subseteq \ker(T - I_V) \Rightarrow \underset{-}{\dim \operatorname{im}(T+I_V)} \le \underset{+}{\dim \ker(T - I_V)}$.

$\therefore \dim \ker(T - I_V) + \dim \ker(T + I_V) \geq \dim V$.

On the other hand, eigenvectors corresponding to 1 and -1 are linearly independent.

$\therefore \dim \ker(T - I_V) + \dim \ker(T + I_V) = \dim V$. $\Rightarrow$ The union of bases of each will be an eigenbasis.

1. **(True/False Jeopardy)** Supply convincing reasoning for your answer.

4

(a) T F Eigenvalues of a matrix may be zero.

(b) T F If two matrices have the same characteristic polynomial, then they must be similar.

(c) T F If $A \in M_{n\times n}$ is invertible and λ is an eigenvalue for A, then $1/\lambda$ is an eigenvalue for A^{-1}.

(d) T F Let $T, S \in \mathcal{L}(V, V)$ have eigenvalues λ and μ, respectively. Then $\lambda + \mu$ is an eigenvalue for $T + S$.

a. True

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

b. False

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \not\sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

c. True

$Av = \lambda v \Rightarrow A^{-1}Av = \lambda \cdot A^{-1}v$

$(\lambda \neq 0) \Rightarrow \frac{1}{\lambda} \cdot v = A^{-1}v$

d. False

$T = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}$ $\quad 0+0=0$ is NOT

$S = \begin{pmatrix} 0 & \\ & 1 \end{pmatrix}$ $\quad$ an eigenvalue of $\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$.