3. Let V denote the set of all solutions to the system of linear equations

$$x_1 - x_2 + 2x_4 = 0$$
$$2x_1 - x_2 - x_3 + 3x_4 = 0.$$

Extend $\{(0,2,1,1)\}$ to a basis of V.

1

Row reductions do everything! We start from $\begin{pmatrix} 1 & -1 & 0 & 2 \\ 2 & -1 & -1 & 3 \end{pmatrix}$.

$$\xrightarrow{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & -1 & -1 \end{pmatrix} \xrightarrow{R_1 \to R_1 + R_2} \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & -1 \end{pmatrix}. \text{ RREF.}$$

As 3rd, 4th col's are non-pivot, let x_3, x_4 be arbitrary real #s.

1st row: $x_1 - x_3 + x_4 = 0 \Rightarrow x_1 = x_3 - x_4$.

2nd row: $x_2 - x_3 - x_4 = 0 \Rightarrow x_2 = x_3 + x_4$.

All together: $\begin{pmatrix} x_1 \\ \vdots \\ x_4 \end{pmatrix} = \begin{pmatrix} x_3 - x_4 \\ x_3 + x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + x_4 \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$

$V = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ and $\dim V = 2$.

$\boxed{\text{Sanity check: } \begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}.}$

So, $\left\{ \begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$ works.

2. Compute the determinant of $A+tI$, where I is the 4×4 identity matrix, $A = \begin{pmatrix} 0 & 0 & 0 & a_0 \\ -1 & 0 & 0 & a_1 \\ 0 & -1 & 0 & a_2 \\ 0 & 0 & -1 & a_3 \end{pmatrix}$, and $t, a_i \in F$ are scalars.

2

① row operations?

$$\det\begin{pmatrix} t & & & a_0 \\ -1 & t & & a_1 \\ & -1 & t & a_2 \\ & & -1 & a_3+t \end{pmatrix} \underset{R_3 \to R_3 + t\cdot R_4}{=} \det\begin{pmatrix} & & & " \\ & & & " \\ 0 & -1 & 0 & (a_2+a_3t+t^2) \\ 0 & 0 & -1 & a_3+t \end{pmatrix} \underset{R_2\to R_2+t\cdot R_3}{=} \det\begin{pmatrix} & & & \\ -1 & 0 & 0 & a_1+a_2t+a_3t^2+t^3 \\ & & & " \\ & & & " \end{pmatrix}$$

$\times t + a_1$

$R_1 \to R_1 + t\cdot R_2$

$$= \det\begin{pmatrix} 0 & 0 & 0 & \boxed{a_0+t\cdot(a_1+a_2t+a_3t^2+t^3)} \\ -1 & 0 & 0 & * \\ 0 & -1 & 0 & * \\ 0 & 0 & -1 & * \end{pmatrix} = a_0+a_1t+a_2t^2+a_3t^3+t^4.$$

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② cofactor expansion : first row

$$\det\begin{pmatrix} t & & & a_0 \\ -1 & t & & a_1 \\ & -1 & t & a_2 \\ & & -1 & a_3+t \end{pmatrix} \quad t\cdot\det\begin{pmatrix} t & 0 & a_1 \\ -1 & t & a_2 \\ 0 & -1 & a_3+t \end{pmatrix} - \underbrace{a_0\det\begin{pmatrix} -1 & t & \\ 0 & -1 & t \\ 0 & 0 & -1 \end{pmatrix}}_{a_0\cdot(-1)^3}$$

For $\det\begin{pmatrix} t & 0 & a_1 \\ -1 & t & a_2 \\ 0 & -1 & a_3+t \end{pmatrix}$ (first row)

$$= t\cdot\det\begin{pmatrix} t & a_2 \\ -1 & a_3+t \end{pmatrix} + a_1\cdot\det\begin{pmatrix} -1 & t \\ 0 & -1 \end{pmatrix}$$

$$= t\cdot(t^2+a_3t+a_2)+a_1$$

$$= t\left[t\cdot(t^2+a_3t+a_2)+a_1\right]+a_0 = t^4+a_3t^3+a_2t^2+a_1t+a_0.$$

1. Let $A, B \in M_{n\times n}(\mathbb{R})$ be matrices such that $AB = -BA$. Prove that if n is odd, then either A or B is not invertible. When n is even, determine whether this is true. If so, give a proof. If not, provide a counterexample.

3 Poll

Fact. ① A is invertible if and only if $\det A \neq 0$.

② $\det(c \cdot A) = c^n \cdot \det(A)$ if $A: n\times n$. & $\det(AB) = \det(A)\det(B)$

Sol'n. As $AB = -BA$, $\det(AB) = \det(-BA) = (-1)^n \cdot \det(BA)$

$\det A \cdot \det B \quad = -1 \cdot \det B \cdot \det A$ b/c n: odd.

So, $\det A \cdot \det B = 0$ and so A or B has det zero $\Leftrightarrow$ A or B is not invertible.

$n=2$ case: try to find conditions for a counterexample!

Suppose A, B : invertible $AB = -BA \Rightarrow B^{-1}AB = -A$. ($A \sim -A$)

$tr(A) = tr(-A) = -tr(A)$.

$\Rightarrow tr(A) = 0$.

Try $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad B^{-1}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ works.

2. Let $B \in M_{n\times n}(F)$ be fixed, and let $T_B : M_{n\times n}(F) \to M_{n\times n}(F)$ be the linear transformation defined by $T_B(A) = AB - BA$. Compute $\det([T_B]_\beta)$, where β is any basis of $M_{n\times n}(F)$.

3 Poll

β does not really matter! (b/c $[T_B]_\alpha = ([I]_\alpha^\beta)^{-1}[T_B]_\beta [I]_\alpha^\beta$)

Fact: T is an isomorphism if and only if $[T]_\beta$ is invertible (for any basis β).

Why? ($\Rightarrow$) T^{-1} exists. So, $T \circ T^{-1} = I$ and

$$[T]_\beta \cdot [T^{-1}]_\beta = [T\circ T^{-1}]_\beta = [I]_\beta = I_{n\times n}.$$

Sol'n. If T_B is not an isom, then $\det([T_B]_\beta) = 0$.

If $\ker(T_B)$ is not zero, then $\det([T_B]_\beta) = 0$. (b/c $T_B : V \to V$).

$\ker(T_B) \ni I_{n\times n} \neq 0$ (b/c $I_{n\times n}B - BI_{n\times n} = B - B = 0$).

1. **(True/False Jeopardy)** Supply convincing reasoning for your answer.

5

(a) T F For $c \in F$ and $A \in M_{n\times n}(F)$, we have $\det(cA) = c\det(A)$.

(b) T F Let $A, B \in M_{n\times n}(F)$. Then $\det(A+B) = \det(A) + \det(B)$.

(c) T F Let V be a finite-dimensional vector space, and let $T : V \to V$ be a linear transformation. If β, β' are two bases of V, then $\det([T]_\beta) = \det([T]_{\beta'})$.

(d) T F Let $A \in M_{m\times n}(F)$. If $m > n$, then there exists some $b \in F^m$ such that $Ax = b$ is inconsistent.

(e) T F Let $A \in M_{m\times n}(F)$. If $m < n$, then there exists some $b \in F^m$ such that $Ax = b$ has a unique solution.

a. FALSE

c^n not c.

b. Use 0 and 1 OR $B = ? A$

FALSE.

$B = A$ $\quad \det(2B) = 2\det(B)$

not true.

c. $[T]_\beta \overset{?}{=} [T]_{\beta'}$?

inverse

$[I]_{\beta'}^{\beta}$ $\quad [I]_{\beta}^{\beta'}$

True.

Similar $\Rightarrow$ det same.

d. A defines a linear map $\mathbb{R}^n \to \mathbb{R}^m$.

$A : F^n \to F^m$ (smaller to larger)

$\operatorname{Im} A \neq F^m$.

True. (b/c by dim. thm. $\dim \ker A + \operatorname{rk} A = n$

$\Rightarrow \operatorname{rk} A \le n < m = \dim F^m$)

e. $Ax = b$ unique sol'n.

What can you say about $Ax = 0$?

$Ax = 0$ has a unique sol'n.

$A : F^n \to F^m$

$\Downarrow$

A is 1-1.

$\ker A = 0$.

2. Let $A \in M_{m\times n}(\mathbb{R})$, and suppose $m > n$. We know that $Ax = b$ does not always have a solution for $b \in \mathbb{R}^m$. Prove that nonetheless, $A^tAx = A^tb$ always has a solution.

4 Math 54 Review

Fun fact: $A^tAx=0$, then in fact $Ax=0$.

Why? ① Note that, given $v=(v_1,\cdots,v_m)\in\mathbb{R}^m$ column vector, $v^t\cdot v$ as a matrix multiplication is simply $v_1^2+\cdots+v_m^2$.

② Now, suppose that $A^tAx=0$. Then, we can multiply x^t on the left to get $x^tA^tAx=x^t\cdot 0=0$. Putting $v=Ax$, we get $v^tv=0$ and this shows that $v_1=\cdots=v_m=0$, so $v=0$ and $Ax=0$.

So what? Obviously, if $x\in\ker A$, then $Ax=0 \Rightarrow A^tAx=0$ $\Rightarrow x\in\ker A^tA$. Combining this with the above "fun fact", we get

$$\ker A = \ker A^tA.$$

But $A:\mathbb{R}^n\to\mathbb{R}^m$ and $A^tA:\mathbb{R}^n\to\mathbb{R}^n$.

Hence, by the dimension theorem, we get $\operatorname{rk} A = n-\dim\ker A$
$= n-\dim\ker A^tA = \operatorname{rk} A^tA$.

However, $\operatorname{im} A^tA \subseteq \operatorname{im} A^t$ obviously, so $\operatorname{rk} A^tA \le \operatorname{rk} A^t$. In particular, $\operatorname{rk} A\le\operatorname{rk} A^t$.

Doing the same argument to A^t, we get $\operatorname{rk} A^t \le \operatorname{rk}(A^t)^t = \operatorname{rk} A$.

Therefore $\operatorname{rk} A = \operatorname{rk} A^tA = \operatorname{rk} A^t$, that is, $\dim\operatorname{im} A^tA = \dim\operatorname{im} A^t$.

Hence, $\operatorname{im} A^tA = \operatorname{im} A^t$. ▨