3. Find ordered bases β and β' of $\mathbb{R}^3$ such that $Q = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 1 & 3 \end{pmatrix}$ is the change of coordinates matrix from β'-coordinates to β-coordinates. Can every invertible matrix be thought of as a change of coordinates matrix in this way?

1

We can maybe let β' or β be $\mathcal{E} = \{e_1, e_2, e_3\}$. Which one sounds easier?

The change of coord. matrix from β' to β : $[I]_{\beta'}^{\beta} = \left([\beta'\text{-vectors}]_{\beta} \right)$

Letting $\beta = \mathcal{E}$ will be simpler b/c $[v]_{\mathcal{E}} = v$.

So, $\beta' = \left\{ \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix} \right\}$ works b/c $[v]_{\mathcal{E}} = v$.

Yes (for the 2nd part) b/c we can let $\beta = \mathcal{E}$ and $\beta' = \{\text{col'n vectors of } Q\}$.

2. Find, with proof, two 2×2 real matrices that are invertible and not similar. Do not use any theorems (or the next problem).

2

Think easy! Our first matrix will be $I_{2\times 2}$ (identity). What's the other?

Suppose A is similar to $I_{2\times2}$. Then $A = Q^{-1} I_{2\times 2} Q$ for some Q: inv. able.
$= Q^{-1}\cdot Q = I_{2\times2}$.

$tr(Q^{-1}AQ) = tr(QQ^{-1}A)$
$= tr(A)$.

$\{ I_{2\times2}$, any other (invertible) 2×2 matrix $\}$ WORKS

$\begin{pmatrix}1&1\\0&1\end{pmatrix}$ or $\begin{pmatrix}2&0\\0&1\end{pmatrix}$ or $(\cdots)$

But $tr(ABC) = tr(BCA)$

3. Show that if $A, B \in M_{n\times n}(F)$, then $tr(AB) = tr(BA)$. Conclude that similar matrices have the same trace. Warning: The "rule" $tr(ABC) = tr(ACB)$ is not true!

2

Poll

Let $A = (a_{ij})$ and $B = (b_{ij})$ and compute tr of AB?

Is this the best?

Observation: if A, B, C satisfies $tr(AB) = tr(BA)$, $tr(AC) = tr(CA)$, then $tr(A(\overset{a}{B}+\overset{b}{C})) = tr((\overset{a}{B}+\overset{b}{C})A)$.

$tr(AB+AC) \quad tr(BA+CA)$

$tr(AB)+tr(AC) = tr(BA)+tr(CA)$

$\Longrightarrow$ The set of matrices B st $tr(AB)=tr(BA)$ is a vector space.

Our goal: It actually is $M_{n\times n}(F)$. (has a "std" basis $E_{ij} = \begin{array}{c} j \\ \begin{pmatrix} \cdots & 1 & \cdots \end{pmatrix} i \end{array}$)

So, it is enough to prove $tr\left(A\cdot\begin{pmatrix}\cdots & 1 & \cdots\end{pmatrix}\right) = tr\left(\begin{pmatrix}\cdots & 1 & \cdots\end{pmatrix}\cdot A\right) \Longleftarrow$ You can check this easily.

5. Let A and B be similar matrices in $M_{n\times n}(F)$. Show that $\text{rank}(A) = \text{rank}(B)$.

2 Poll

You could consider im A and compare this with B.
But we will use ker A instead b/c it's easier to deal with.

Claim. If A is similar to B ($A = Q^{-1}BQ$), then dim ker $A \le$ dim ker B.

Why is this enough? ① So B is sim. to $A \Rightarrow$ dim ker $B \le$ dim ker A.
So, dim ker A = dim ker B.
② By Dim_Thm., rk $A = n -$ dim ker $A = n -$ dim ker $B =$ rk B.

Pf. $A = Q^{-1}BQ$ so $QA = BQ$.
If $v \in \ker A$, then $Av = 0$ so $QAv = 0$ so $BQv = 0$ so $Qv \in \ker B$.

We can define a lin. trans $T_Q : \ker A \to \ker B$
$v \longmapsto Q \cdot v$.

This 1-1 b/c if $Q \cdot v = 0$ then by left multiply Q^{-1} we get $v = 0$.

$\Longrightarrow$ dim ker $A \le$ dim ker B.

4

1. **(True/False Jeopardy)** Supply convincing reasoning for your answer.

(a) T F If A is similar to B and A is invertible, then B is also invertible.

(b) T F The set of solutions to a homogenous system of linear equations is a vector space.

(c) T F If $A \in M_{n\times n}$ is upper triangular and B is similar to A, then B is also upper triangular.

(d) T F If two $n \times n$ matrices have the same trace, then they are similar.

a) $A = Q^{-1}BQ$

True b/c

$B = QAQ^{-1}$

and $QA^{-1}Q^{-1}$ is the inverse B^{-1}!

c) Use 0 & 1 only

$Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$Q^{-1}AQ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

d) Recall a previous problem

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

2. Let $T : V \to W$ be a linear transformation. Show that if T is surjective, then $T^* : W^* \to V^*$ is injective. Conclude that if the columns of an $n \times m$ matrix generate $\mathbb{R}^n$, then its rows must be linearly independent.

3

① T^* is linear. (Checked in class)

② Now, it is enough to consider $\ker T^*$.

$$\begin{aligned}
\ker T^* &= \{g \in W^* : T^*(g) = 0_{V^*} \in V^*\} \\
&= \{g \in W^* : T^*(g)(v) = 0 \text{ for all } v \in V\} \\
&= \{g \in W^* : g(T(v)) = 0 \text{ for all } v \in V\} \\
&= \{g \in W^* : g(w) = 0 \text{ for all } w \in W\} \\
&= \{0_{W^*}\}.
\end{aligned}$$

$\operatorname{im} T = W.$

③ Given an $n \times m$ matrix M. consider $T : \mathbb{R}^m \to \mathbb{R}^n$ defined by $T(v) = M \cdot v$.

Then "columns of M generate $\mathbb{R}^n$" $\Longleftrightarrow$ "$T(\mathbb{R}^m)$ spans $\mathbb{R}^n$"

(Note that $M = [T]_{\mathcal{E}_m}^{\mathcal{E}_n}$.) $\Longleftrightarrow$ "$\operatorname{im} T = \mathbb{R}^n$" $\Longleftrightarrow$ T is surjective.

Hence, T^* is injective. So, $[T^*]_{\mathcal{E}_n^*}^{\mathcal{E}_n^*}$ has linearly independent columns.

However, we have $[T^*]_{\mathcal{E}_n^*}^{\mathcal{E}_m^*} = \left([T]_{\mathcal{E}_m}^{\mathcal{E}_n}\right)^t$, so M's rows are linearly independent.