1. Let $V = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in M_{2\times 2}(\mathbb{R}) \mid a, b \in \mathbb{R} \right\}$, and let $W = \mathbb{C}$ considered as vector spaces over $\mathbb{R}$. Prove that $V \cong W$. $= \mathrm{Span}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$ 2-dim'l

Poll

1

"$\cong$" means "isomorphic". How do you prove "$\cong$"? Find an isomorphism T.

$T: V \to W$ sends $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ to $\boxed{a+bi}$

linearity (exercise)

1) T is one-to-one.

$$\ker T = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a+bi = 0 \right\}$$
$$= \{ \ \text{''} \ : a=0, b=0 \}$$
$$= \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} = \{\vec{0}\}.$$

2) T is onto.

dimension theorem

$\Rightarrow \dim \ker T + \dim \operatorname{im} T = \dim V = 2$, with $\dim \ker T = 0$

$\Rightarrow \dim \operatorname{im} T = 2$

$\Rightarrow \operatorname{im} T = W$.

Remark 1. Another T?

$T'\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = a - bi$

$a + 2bi$

$b + ai$

$7b + 3ai$

A lot!

Remark 2.

Linear Algebra Expert will say

$\dim V = \dim W$, so

$V \cong W$.

Interesting Fact: T is not only linear!

$$T\left(\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} c & d \\ -d & c \end{pmatrix} \right) = T\begin{pmatrix} a & b \\ -b & a \end{pmatrix} T\begin{pmatrix} c & d \\ -d & c \end{pmatrix}$$

$$ac - bd + (ad + bc)i = (a+bi)(c+di)$$

MATH 113 deals with "multiplication" operation!!!

2. Let F be a field, and let A be a matrix such that $A^n = 0$ for some $n > 0$. Prove that $I - A$ is invertible. Bonus challenge: find the inverse of $I - A$.

$m \times m$

3

$I-A$: invertible $\iff \ker(I-A) = \{\vec{0}\}$.
(= isomorphism (lin. trans.)) $\quad (\mathrm{Nul}(I-A))$

$\ker(I-A) = \{v \in \mathbb{R}^m : (I-A)\cdot v = 0\}$

$= \{v \in \mathbb{R}^m : Av = v\}$.

$\vdots$

$= \{\vec{0}\}$

Let's consider

$A^2 v = A \cdot A \cdot v$
$\quad = A \cdot v = v$.

$A^3 v = A \cdot A^2 \cdot v$
$\quad = A \cdot v = v$.

$\vdots$

$A^n v = v$. for any n.

$\Rightarrow$ (for that specific n) $\underset{\substack{\| \\ 0}}{0 \cdot v} = v$.

You, a math expert, CAN just come up with:

$(1-x)(1+x+x^2+x^3+x^4+\cdots+x^n) = 1-x^{n+1}$.

$(I-A)\underbrace{(I+A+A^2+\cdots+A^{n-1})}_{\text{the inverse of } I-A} = I - A^n = I$.

4

(a) T F Let F be a field. Then F^n is isomorphic to F^m if and only if $n = m$.

(b) T F Let F be a field. Then $M_{n \times m}(F)$ and $M_{n' \times m'}(F)$ are isomorphic if and only if $n = n'$ and $m = m'$.

(a) ($\Leftarrow$): Easy (ex. $T = ?$)

($\Rightarrow$): There exists an isom.

$T : F^n \to F^m$. Look at a basis!

T(basis vectors) becomes basis vectors

(b) BE LAZY!

column $\cong$ row.

$n \times 1$ $\xleftarrow{\text{transpose}}$ $1 \times n$.

(c) T F If $T : V \to W$ and $U : W \to V$ are linear transformations such that $TU = \mathrm{id}_W$, then $UT = \mathrm{id}_V$.

(d) T F If $T, S \in \mathcal{L}(V, W)$ are isomorphisms, then $T + S$ is also an isomorphism.

(e) T F If $A, B \in M_{n\times n}(F)$ are invertible and $AB = BA$, then $A^{-1}B^{-1} = B^{-1}A^{-1}$.

Poll

4

(c) $\dim V = \dim W \Rightarrow$ True.

False ex.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(e) True.

$$\begin{array}{ccc} (AB)^{-1} & = & (BA)^{-1} \\ \| & & \| \\ B^{-1}A^{-1} & & A^{-1}B^{-1} \end{array}$$

(d) LAZY Strategy FALSE.

$V = W = \mathbb{R}^2$

$T = I_{2\times 2}$, $S = -I_{2\times 2}$.

$T + S = O$.