

2. Let F be a field, and let A be a matrix such that $A^n = 0$ for some $n > 0$. Prove that $I - A$ is invertible. Bonus challenge: find the inverse of $I - A$.

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$I - A$: invertible $\iff \ker(I - A) = \{ \vec{0} \}$.
 (= isomorphism (in. trans.)) (Nul($I - A$))

$$\ker(I - A) = \{ v \in \mathbb{R}^m : (I - A) \cdot v = \vec{0} \}$$

$$= \{ v \in \mathbb{R}^m : Av = v \}$$

$$\vdots$$

$$= \{ \vec{0} \}$$

Let's consider

$$A^2 v = A \cdot A \cdot v$$

$$= A \cdot v = v.$$

$$A^3 v = A \cdot A^2 \cdot v$$

$$= A \cdot v = v.$$

$$\vdots$$

$$A^n v = v. \text{ for any } n.$$

$$\Rightarrow \text{(for that specific } n) \quad \vec{0} \cdot v = v.$$

You, a math expert, CAN just come up with:

$$(1 - x)(1 + x + x^2 + x^3 + \dots + x^{n-1}) = 1 - x^n$$

$$(I - A)(I + A + A^2 + \dots + A^{n-1}) = I - A^n = I.$$

the inverse of $I - A$.

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(a) T F Let F be a field. Then F^n is isomorphic to F^m if and only if $n = m$.

(b) T F Let F be a field. Then $M_{n \times m}(F)$ and $M_{n' \times m'}(F)$ are isomorphic if and only if $n = n'$ and $m = m'$.

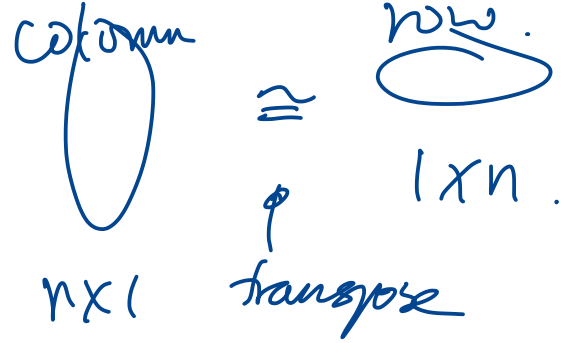
(a) (\Leftarrow): Easy (ex. $T = ?$)

(\Rightarrow): There exists an isom.

$T: F^n \rightarrow F^m$. Look at a basis!

$T(\text{basis vectors})$ becomes basis vectors

(b) BE LAZY!



Poll

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(c) T F If $T : V \rightarrow W$ and $U : W \rightarrow V$ are linear transformations such that $TU = \text{id}_W$, then $UT = \text{id}_V$.

(d) T F If $T, S \in \mathcal{L}(V, W)$ are isomorphisms, then $T + S$ is also an isomorphism.

(e) T F If $A, B \in M_{n \times n}(F)$ are invertible and $AB = BA$, then $A^{-1}B^{-1} = B^{-1}A^{-1}$.

(c) $\dim V = \dim W \Rightarrow \text{True}$.

False
ex. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(e) True.

$$(AB)^{-1} = (BA)^{-1}$$

$$\begin{matrix} \parallel & \parallel \\ B^{-1}A^{-1} & A^{-1}B^{-1} \end{matrix}$$

(d) LAZY Strategy FALSE.

$$V = W = \mathbb{R}^2$$

$$T = I_{2 \times 2}, S = -I_{2 \times 2}$$

$$T + S = \mathbf{0}$$