1. Let $\beta = \{1, x, x^2\}$ be the standard basis for $\mathbb{P}_2(\mathbb{R})$ and let $\gamma = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ be the standard basis for $\mathbb{R}^2$. Consider the linear transformation $T : \mathbb{P}_2(\mathbb{R}) \to \mathbb{R}^2$ given by $T(p) = \begin{pmatrix} p(1) \\ p(2) \end{pmatrix}$. Find $[T]_\beta^\gamma$.

3-dim'l   2-dim'l   1   Poll

$\underset{m}{3} \rightarrow \underset{n}{2}$

How do you find $[T]_\beta^\gamma$? Apply β's vectors and read it with respect to γ's vectors.

$[T]_\beta^\gamma = \Big([T(v_1)]_\gamma \cdots [T(v_n)]_\gamma \Big)$ where $\beta = \{v_1, \ldots, v_n\}$.

3-columns

this prob.

$= \Big([T(1)]_\gamma \; [T(x)]_\gamma \; [T(x^2)]_\gamma \Big)$

$= \left(\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]_\gamma \; \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right]_\gamma \; \left[\begin{pmatrix} 1 \\ 4 \end{pmatrix}\right]_\gamma \right)$ 2-rows   3

$= 2\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix}.$

2×3

$n \times m$

Luckily, γ is the std one.

$\Rightarrow \left[\begin{pmatrix} a \\ b \end{pmatrix}\right]_\gamma = \begin{pmatrix} a \\ b \end{pmatrix}$

What happens if $\gamma = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$

$\left[\begin{pmatrix} a \\ b \end{pmatrix}\right]_\gamma = ?$ $\begin{pmatrix} b \\ a \end{pmatrix}$

$\Uparrow$

$\begin{pmatrix} a \\ b \end{pmatrix} = b \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} + a \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

if γ was $\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$, then

$[T]_\beta^\gamma = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 1 & 1 \end{pmatrix}$

2. Let $\beta = \{v_1, v_2, v_3\}$ and let $\gamma = \{w_1, w_2\}$ be bases for vector spaces V and W, respectively, over a field F. Suppose that $T \in \mathcal{L}(V)$ is given by the matrix $[T]_\beta^{\beta\,\gamma} = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \end{pmatrix}$. Find a nonzero vector in $\ker(T)$.

typo : $T \in \mathcal{L}(V, W)$

1

$\ker T = \{ v \in V : T(v) = \vec{0}_W \}$.

(Honest way)

- $[T]_\beta^\gamma$ is the unique matrix satisfying $[T(v)]_\gamma = [T]_\beta^\gamma \cdot [v]_\beta$.

---

if $v \in \ker T$, = then $A \cdot X$ = b

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$\xrightarrow{\text{r.r.}}$ a.m. . . . . .

x: free; $y = z = -x$

just let $x = 1$. Then, $[v]_\beta = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ will work.

$v = v_1 - v_2 - v_3$ works.

(Being greedy)

$\ker T \sim$ "$T(v) = 0$"

consider $\{v_1, v_2, v_3\}$: a basis

$v = a_1 v_1 + a_2 v_2 + a_3 v_3$.

$\sim$ "$a_1 T(v_1) + a_2 T(v_2) + a_3 T(v_3) = 0$."

find a_1, a_2, a_3.

$\sim$ find a lin. relation betw three columns.

$$a_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + a_3 \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow a_2 = a_3 = 1$ & just let

$a_1 = -1$

$(a_1, a_2, a_3) = (-1, 1, 1)$

So, (probably) $-v_1 + v_2 + v_3$ works.

Check $T(-v_1 + v_2 + v_3) = \vec{0}_W$.

1. Define $T, S \in \mathcal{L}(\mathbb{P}_2(\mathbb{R}), \mathbb{P}_2(\mathbb{R}))$ by $T(p) = p'$ and $S(p) = xp'$. Compute the matrices of T and S and $S \circ T$ separately with respect to the basis $\beta = \{1, x, x^2\}$ and verify that $[S \circ T]^\beta_\beta = [S]^\beta_\beta [T]^\beta_\beta$ in this example. Here p' denotes the derivative of p and all polynomials are in the variable x.

2

Composition of functions: $f(x) = x^2$, $g(x) = x+2$

$\Rightarrow (f \circ g)(x) = ?$

$$(f \circ g)(x) = f(g(x)) = f(x+2) = (x+2)^2$$

What is $(S \circ T)(p) = ?$

$$(S \circ T)(p) = S(T(p)) = S(p') = x \cdot (p')' = x \cdot p''.$$

$$[T]^\beta_\beta = \left([T(1)]_\beta \; [T(x)]_\beta \; [T(x^2)]_\beta \right) = \left([0]_\beta \; [1]_\beta \; [2x]_\beta \right)$$

$0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2$

$1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2$

$2x = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

CAUTION

$$\cancel{\Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}}$$

$$[S]^\beta_\beta = \left([S(1)]_\beta \; [S(x)]_\beta \; [S(x^2)]_\beta \right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$[S \circ T]^\beta_\beta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

!?

$$\cancel{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}} = \begin{pmatrix} * & 1 & * \\ * & * & * \\ * & * & * \end{pmatrix}$$

Comments on the ORDERED basis:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \overset{?}{=} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Exercise.

1. **(True/False Jeopardy)** Supply convincing reasoning for your answer.

(a) T F Let V be a vector space of dimension 2 and let $T \in \mathcal{L}(V, V)$. If $[T]_\beta^\beta = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ for some basis β of V, then $[T]_\beta^\beta = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ for every basis β of V.

(b) T F Let A, B, C be $n \times n$ matrices over F such that $AB = BA$ and $BC = CB$. Then $AC = CA$.

(c) T F If $A, B \in M_{n\times n}(F)$, then $(A+B)^2 = A^2 + 2AB + B^2$.

Poll

4

Notification.
"tech-practice" midterm on Sunday 6pm–9pm. Check out the Webwork tab on bCourses.

(a) Compare with the previous problem.

In general, it should be false.

But, here it's true:

Let $\beta = \{v_1, v_2\}$, then

$T(v_1) = 2\cdot v_1 + 0\cdot v_2 = 2v_1$

$T(v_2) = 0\cdot v_1 + 2\cdot v_2 = 2v_2$

$\Rightarrow T(v) = 2v \quad \forall v \in V$

(But, this is the only case basically)

$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

(b) Be LAZY!

Make this to nothing!

$B = 0.$

(c) Make use of (b).

$(A+B)^2 = (A+B)\cdot(A+B)$

$= (A+B)\cdot A + (A+B)\cdot B$

$= A^2 + BA + AB + B^2.$

We saw that BA is not necessarily the same as AB

(ex: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$)

False.