2. Let S be as in the previous problem, and let $B = \{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}$ be the standard basis of $\mathbb{R}^4$. Extend your basis from the first problem to a basis of $\mathbb{R}^4$ by adding vectors from B.

e_1, e_2, e_3, e_4

1 Poll.

• $S = \{(1,2,3,4),(2,3,4,5),(3,4,5,6)\}$.
Compute dim span(S) and find a basis for span(S).

1) Which vector is not necessary in S? (previous problem)

$a \cdot \underset{v_1}{(1,2,3,4)} + (-2)\,b \cdot \underset{v_2}{(2,3,4,5)} + c \cdot \underset{v_3}{(3,4,5,6)} = \vec{0}$.

$v_3 = -v_1 + 2v_2 \Rightarrow v_3$ is "dependent upon" v_1 & v_2.

$\{v_1, v_2\}$: lin. indep? $v_2 \neq c \cdot v_1$ for any $c \in \mathbb{R} \Rightarrow$ Yes it is.

2) Which vector in B is "new" for S?

Check if your choice of vectors in B is "independent" with $\{v_1, v_2\}$.

$\{e_1, v_1, v_2\}$: lin. indep? Yes, it is.

$\underline{\{e_1, e_2, v_1, v_2\}}$: lin. indep? " .

Answer.

1. Let V be a vector space over F with basis $B = \{v_1, \dots, v_n\}$. Let W be a vector space over F, and let $w_1, \dots, w_n \in W$ be arbitrary elements (not necessarily distinct). Prove that there exists a unique linear transformation T such that $T(v_i) = w_i$ for all i.

2

- Existence (sometimes "well-definedness" is the key.)

Given $v \in V$, we have $c_1, \dots, c_n \in F$ s.t. $v = c_1 v_1 + \cdots + c_n v_n$.

Define $T(v)$ to be $c_1 w_1 + \cdots + c_n w_n$.

v is uniquely expressed in this way. T is well-defined.

$\boxed{T(au+bv) = aT(u) + bT(v) \quad \text{(Linearity)}}$

$\hookrightarrow$ Why? (example)

$V = \mathbb{R}^2$, $S = \{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\}$

Want to define lin. T s.t. $T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$

$T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\2\end{pmatrix}$

$T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\\cancel{1}\ 2\end{pmatrix}$.

Fails to exist.

$T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$

$\| \qquad \qquad \#$

$T\left(\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\right) = \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\2\end{pmatrix}$

- Uniqueness

Let T' be a lin. transformation satisfying the conditions

For any $v \in V$, we have $a_1, \dots, a_n \in F$ s.t. $v = a_1 v_1 + \cdots + a_n v_n$.

Then $T'(v) = T'(a_1 v_1 + \cdots + a_n v_n)$

$= a_1 T'(v_1) + \cdots + a_n T'(v_n)$ } linearity

$= a_1 w_1 + \cdots + a_n w_n$ } conditions

$= T(v)$.

1. **(True/False Jeopardy)** Supply convincing reasoning for your answer.

$\#P(F)$ $\quad$ $\mathbb{F}_2$ $\quad$ 4 Poll.

(a) T F Let F be a field. Then $\mathcal{F}(F, F)$ is infinite-dimensional.

(b) T F Let V be a vector space of dimension n, and let S be a linearly dependent subset of V with $\#S > n$. Then there exists a subset $B \subseteq S$ which is a basis for V.

(c) T F If $V = U \oplus W$, and S is a basis of V such that some subset $S_U \subseteq S$ is a basis of U, then $S \setminus S_U$ is a basis of W.

(d) T F Let V be a vector space of dimension $n \geq 1$, and let S be a subset of V with $\#S < n$. Then S is linearly dependent.

(e) T F Suppose that $V = U \oplus U'$, and let $T : V \to W$ be a linear transformation. Then $\mathrm{Im}(V) = \mathrm{Im}(U) \oplus \mathrm{Im}(U')$.

$\mathcal{F}(\mathbb{R}, \mathbb{R})$: inf. dim'l.

$\{\sin x, \sin 2x, \dots, \dots\}$

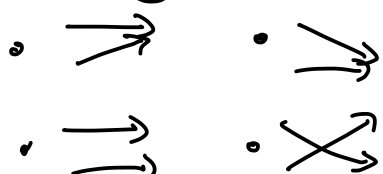
$$\underline{f(x) = x^2 - x}$$
$$\neq 0$$

(a) $F = \mathbb{F}_2$ (or $\mathbb{F}_3, \dots$)

$\Downarrow$

$\mathcal{F}(F, F)$

$f : \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\quad \begin{pmatrix} a \\ b \end{pmatrix}$ $\quad 2^2$



(b) False.

$V = \mathbb{R}^2$, $n = 2$.

$S = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \end{pmatrix} \right\}$

$\#S > 2$.

(d) False.

$S = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ in $\mathbb{R}^2$

$n = 2$.