2. (a) Let $U = \{f \in \mathbb{P}_3(\mathbb{R}) : f(0) = 0\}$. Find a set of polynomials in $\mathbb{P}_3(\mathbb{R})$ whose span is U. You do not need to check that U is a subspace.

(b) Do the same for the subspace $W = \{f \in \mathbb{P}_3(\mathbb{R}) : f'(0) = 0\}$.

Poll.

1

(a) • $\{ax^3+bx^2+cx : a,b,c \in \mathbb{R}\}$ works. because it is just U.

• $\{x, x^2, x^3\}$ also works.

(b) If $f(x) = ax^3+bx^2+cx+d$, then "$f'(0)=0$" means "$3ax^2+2bx+c\big|_{x=0}=0$".

$\Rightarrow c = 0$.

$W = \{ax^3+bx^2+d : a,b,d \in \mathbb{R}\}$.

• $\{ax^3+bx^2+d\}$ works.

• $\{x^3, x^2, 1\}$ works.

1

3. Let $\mathcal{A} = \{A \in M_{2\times2}(\mathbb{R}) : A^t = -A\}$ (← anti-symmetric) and let $\mathcal{B} = \{B \in M_{2\times2}(\mathbb{R}) : B$ is upper triangular$\}$. These are both subspaces of $M_{2\times2}(\mathbb{R})$. Show that $M_{2\times2}(\mathbb{R}) = \mathcal{A} + \mathcal{B}$. Is the sum direct?

$M_{2\times2}(\mathbb{R}) = \mathcal{A} + \mathcal{B}$

In general, proving $A = B$ "=" proving $A \subseteq B$ and $B \subseteq A$.

And "proving $A \subseteq B$" is "for any element $x \in A$, x belongs to B".

• $\mathcal{A} \subsetneq M_{2\times2}$, $\mathcal{B} \subsetneq M_{2\times2}$, so trivially, $\mathcal{A}+\mathcal{B} \subseteq M_{2\times2}$.

• We need to prove that $M_{2\times2} \subseteq \mathcal{A}+\mathcal{B}$.

Let M be an element of $M_{2\times2}(\mathbb{R})$ $\left(M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$.

$$M = \underbrace{\begin{pmatrix} 0 & -c \\ c & 0 \end{pmatrix}}_{\in \mathcal{A}} + \underbrace{\begin{pmatrix} a & (b+c) \\ 0 & d \end{pmatrix}}_{\in \mathcal{B}}.$$

So, $M \in \mathcal{A}+\mathcal{B}$. $\Rightarrow M_{2\times2} \subseteq \mathcal{A}+\mathcal{B}$.

$\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$: the form of $\mathcal{B}$'s elements

$\begin{pmatrix} 0 & -w \\ w & 0 \end{pmatrix}$: the form of $\mathcal{A}$'s elements

1. Suppose that $V = U \oplus W$. Show that every $v \in V$ can be *uniquely* written in the form $v = u + w$ with $u \in U$ and $w \in W$.

2

- Existence

$v \in V = U \oplus W$,

so $\exists u \in U, w \in W$

s.t. $v = u + w$.

- Uniqueness.

Suppose that $v = u + w$ for some $u \in U$, $w \in W$

and $v = u' + w'$ for some $u' \in U$, $w' \in W$.

Then $u + w = u' + w'$. So, $u - u' = w' - w \in W$.

$u - u' \in U$ (b/c U is a vec. sp.)

As it is direct, $U \cap W = \{\vec{0}\}$.

$\Rightarrow u - u' = \vec{0}$, $w' - w = \vec{0}$,

Goal: $u = u'$, $w = w'$.

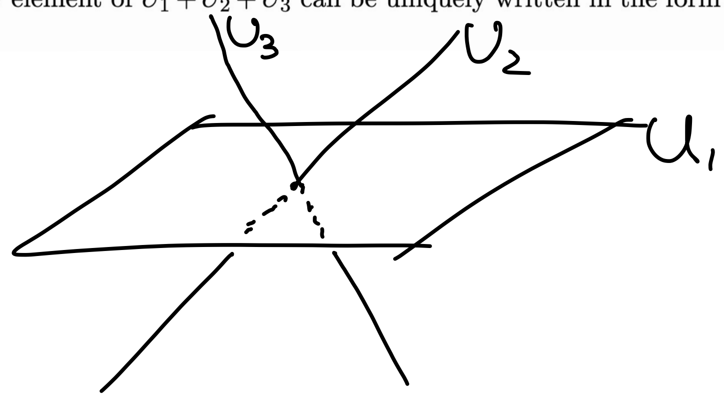
5

2. Earlier, you saw that if U_1 and U_2 are subspaces of V and $U_1 \cap U_2 = \{0\}$, then every element of $U_1 + U_2$ can be uniquely written in the form $x_1 + x_2$ for $x_1 \in U_1$ and $x_2 \in U_2$.

$U_i \cap (U_j + U_k) = \{\vec{0}\}$

Now suppose that U_1, U_2, U_3 are subspaces such that $U_i \cap U_j = \{0\}$ for $1 \le i < j \le 3$. Is it still true that every element of $U_1 + U_2 + U_3$ can be uniquely written in the form $x_1 + x_2 + x_3$ for $x_i \in U_i$?

not enough



1. Show that the functions $f(x) = \sin(x)$ and $g(x) = \cos(x)$ are linearly independent elements of $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

3

Suppose $a \cdot \sin x + b \cdot \cos x = 0$ as an element of $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

We need to show that $a = 0$, $b = 0$.

for any $x \in \mathbb{R}$, this should be true.

$x = 0 \Rightarrow b = 0$

$x = \pi/2 \Rightarrow a = 0$.

$f : \{0\} \to \mathbb{R}$

$f(x) = 5^x + 1$

$= 2x + 2$.

Poll.

1. **(True/False Jeopardy)** Supply convincing reasoning for your answer.

6

(a) T F A set of vectors S in a vector space is linearly dependent if and only if **each** vector in S can be written as a linear combination of the others.

(b) T F Any two elements of $\mathbb{R}$ are linearly dependent (where $\mathbb{R}$ is considered as a vector space over the field $\mathbb{R}$).

(c) T F If $S \subseteq S'$, then $\mathrm{Span}(S) \subseteq \mathrm{Span}(S')$.

(d) T F If S and S' are finite subsets of a vector space V and $\mathrm{Span}(S) = \mathrm{Span}(S')$, then there are the same number of elements in S and S'.